Polynomial Hierarchy (PH)

Recall

$A \in NP \Rightarrow \quad x \in A \Leftrightarrow \exists \underset{\text{poly length}}{y} : \overbrace{M(x,y)=1}^{\in P}$ $\quad m = n^{O(1)}$

$B \in PSPACE \Rightarrow \quad x \in B \Leftrightarrow \exists y_1 \forall y_2 \exists y_3 \cdots Q y_m : \underbrace{M(x,y)}_{\in P}$

Introduce hierarchy

$P \subseteq NP \subseteq \cdots\cdots \subseteq PSPACE$

Def: $L \in \Sigma_i P \quad i \geq 0$

$x \in L \Leftrightarrow \exists y_1 \forall y_2 \cdots \overset{\exists/\forall}{Q} y_i : \overbrace{M(x,y)=1}^{\in P}$

Def $\Pi_i P = co\Sigma_i P \Rightarrow x \in L \Leftrightarrow \forall y_1 \exists y_2 \cdots Q y_i : M(x,y) = 1$

Note $\Sigma_i P \subseteq \Sigma_{i+1} P$

$\Sigma_i P \subseteq \Pi_{i+1} P$

$NP = \Sigma_1 P$, $\quad \Sigma_0 P = P = \Pi_1 P$, $\quad \Pi_1 P = coNP$

$$\begin{tikzcd}
 & \Sigma_1 P \arrow[r] \arrow[dr] & \Sigma_2 P \arrow[r] \arrow[dr] & \Sigma_i P \arrow[dr] & & \\
P \arrow[ur] \arrow[dr] & & & & PH \arrow[r] & PSPACE \\
 & \Pi_1 P \arrow[r] \arrow[ur] & \Pi_2 P \arrow[r] \arrow[ur] & \Pi_i P \arrow[ur] & &
\end{tikzcd}$$

Def $PH = \bigcup_{i \geq 0} \Sigma_i P \quad \left(= \bigcup_i \Pi_i P \right)$

Ex MinCircuit $= \{ \langle \overset{ckt}{C} \rangle :$ C is a cicuit of minimal size (no smaller ckt $\}$

computing C)

$$\langle C\rangle \in MC \iff \forall C', |C'| < |C| : \exists x : C'(x) \neq C(x)$$

$$\iff \underbrace{\forall C' \exists x : \underbrace{(|C'| < |C|) \Rightarrow C'(x) \neq C(x)}_{\in P}}_{\in \Pi_2 P}$$

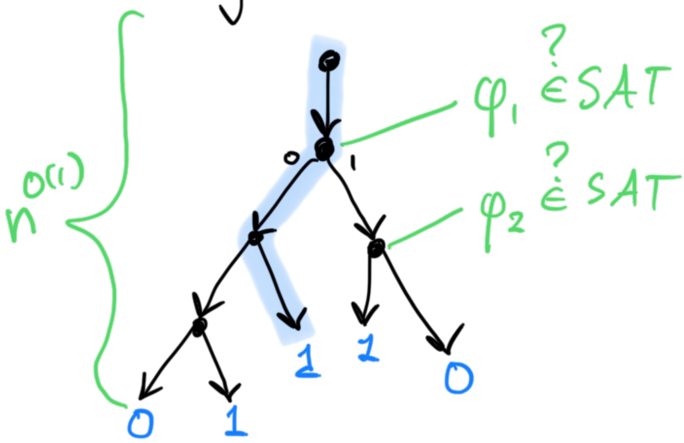
Ex: $\mathrm{MaxIS} = \{\langle G, k\rangle :$ largest IS in G is size $= k\}$

$\mathrm{MaxIS} \in P^{SAT}$, $\mathrm{MaxIS} \in \Sigma_2 P \cap \Pi_2 P$

$$\langle G, k\rangle \in MIS \iff \exists S \subseteq V, \forall S' \subseteq V : \boxed{\begin{matrix} S \text{ is IS } \& \ |S| = k \\ (S' \text{ is IS} \Rightarrow |S'| \leq k) \end{matrix}} \in P$$

Claim: $\underbrace{P^{NP} \subseteq \Sigma_2 P \cap \Pi_2 P}_{\text{closed under complement}}$ $\quad (P^{NP} := P^{SAT})$

Config tree of P^{NP} computation on input x



Accepts x iff

$\exists$ 1-path with answers $(0, 1, 0, \ldots, 1, 0)$

N   Y

$\exists z_i, \quad i \in Y$

$\forall z_i, \quad i \in N$

$\left[\varphi_i(z_i) = 1 \quad \forall i \in Y \right.$

$$P^{NP} \subseteq \Sigma_2 P$$

$$P^{NP} = coP^{NP} \subseteq co\Sigma_2 P = \Pi_2 P \qquad [\varphi_i(z_i) = 0 \quad \forall i \in N]$$

$\uparrow P$

$$\begin{tikzcd}
& NP \arrow[dr] & & \Sigma_2 P \arrow[dr] & & \Sigma_3 P \arrow[dr] & \\
P \arrow[ur] \arrow[dr] & & P^{NP} \arrow[ur] \arrow[dr] \arrow[d, equal] & & P^{\Sigma_2 P} \arrow[ur] \arrow[dr] \arrow[d, equal] & & PH \\
& coNP \arrow[ur] & (\Delta_1 P) & \Pi_2 P \arrow[ur] & (\Delta_2 P) & \Pi_3 P \arrow[ur] &
\end{tikzcd}$$

Popular hypothesis "PH is infinite/strict"

i.e. $PH \neq \Sigma_i P$ for any i

Lemma: $\Sigma_i P = \Pi_i P \Rightarrow$ PH collapses: $PH = \Sigma_i P$

Prove it for $i=1$: $NP = coNP$: Assume $NP = coNP$

Take $L \in PH$. Then $L \in \Sigma_i P$ for some i.

$$x \in L \Leftrightarrow \exists y_1 \forall y_2 \cdots \forall y_{k-1} \underbrace{\exists y_k : \overbrace{M(x,y)=1}^{\in P}}$$

$\{(x, y_1, \dots, y_{k-1}) : \exists y_k : M(x,y)=1\} \in NP = coNP$

$$\Leftrightarrow \exists y_1 \forall y_2 \cdots \underbrace{\forall y_{k-1} \forall y_k : M'(x,y) = 1}_{\text{1 less alternation}}$$

$\Rightarrow L \in \Sigma_{i-1} P$

(iterate)

$\vdots$

$L \in NP$

Thm: $\exists$ oracle A: PH^A is infinite
[80s]

Lemma: PH has no complete problem
(unless PH collapses) [Exercise]

Karp-Lipton $NP \subseteq P/_{poly} \Rightarrow \Sigma_2 P = \Pi_2 P$

Proof Suppose $SAT \in P/_{poly}$

Then $\exists$ poly-size circuits $\{C_n\}_{n\in\mathbb{N}}$ s.t

$\varphi \in SAT \iff C_n(\varphi)$ outputs sat assignment

$\iff \varphi(C_n(\varphi)) = 1$

Let $L \in \Pi_2 P$

Cook-Levin

$$x \in L \iff \forall y\ \exists z : \overbrace{\underbrace{\varphi(x,y,z)=1}_{\varphi_{x,y}(z)}}^{\text{Encoding TM as SAT}}$$

$\Updownarrow$

$$\exists C\ \forall y : \underbrace{\varphi_{x,y}(C(\varphi_{x,y})) = 1}$$

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